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The stabilisation of explosive instabilities in the presence of a third-order nonlinear effect

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Abstract. The interaction of three waves in the presence of a third-order nonlinear interaction term is investigated by the method of nonlinear perturbation. The analytical solutions obtained here are found to be in agreement with the numerical solutions already given by Weiland and Wilhelmsson, and thus complement their work.

1. Introduction

This is a sequel to our previous paper (Khan *et al* 1980), where we discussed within the framework of nonlinear perturbation theory (Coffey and Ford 1969) how the nonlinear three-wave interaction becomes explosive in the presence of linear damping of the waves. It was shown that in the general case when the coupling coefficients become complex with their phases not equal to zero or π, the problem of the occurrence of an explosive instability becomes considerably more complicated. It has been noted that the explosive instability studied by a well defined phase approach is a first-order phenomenon, and this type of instability may be developed in higher orders of nonlinear perturbation.

Recently, Fukai *et al* (1969, 1970), Byers *et al* (1971) and Oraevskii *et al* (1973a,b) have studied the influence of third-order nonlinear terms on explosive instabilities in the coherent phase description for real second-order coupling factors. An upper limit for the amplitude of a nonlinear instability as a result of an amplitude-dependent frequency shift was also pointed out. Nonlinear stabilisation of explosive flute instabilities of mirror confined plasma has been discussed by Dum and Sudan (1969). Weiland and Wilhelmsson (1973) and Weiland (1974) have extended the investigation to include a linear dissipation and also an imaginary part of the third-order frequency shift. In these papers the saturation of the explosive instability by third-order terms is studied analytically and by means of computers. Nonlinear instabilities arising from the interaction of positive and negative energy waves have recently been observed on a computer model (Byers *et al* 1971, Shchinov *et al* 1973). The influence of an explosive instability on the plasma distribution has also been studied (Hamasaki and Krall 1971). Wilhelmsson (1970, 1972) has shown that the imaginary part of the complex third-order coupling factors might have a decisive influence for the stabilisation of explosive instabilities.

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In our present paper the effect of nonlinear dissipation on explosive instabilities has been considered in the presence of second-order coupling factors. It is observed that the amplitudes of the waves lead to stabilisation in the presence of such dissipation proportional to the square of the amplitudes, but in the absence of any linear dissipation the amplitudes do exhibit unlimited growth.

2. Basic equations

The basic equations of the waves interacting nonlinearly to higher orders are

$$
\begin{aligned}
\frac{\partial a_0}{\partial t} - \mathrm{i}\omega_0 a_0 &= c_{12}^* a_1 a_2 - \mathrm{i} a_0 \sum_{k=0}^{2} \alpha_{0k} |a_k|^2, \\
\frac{\partial a_1}{\partial t} - \mathrm{i}\omega_1 a_1 &= c_{02} a_0 a_2^* - \mathrm{i} a_1 \sum_{k=0}^{2} \alpha_{1k} |a_k|^2, \\
\frac{\partial a_2}{\partial t} - \mathrm{i}\omega_2 a_2 &= c_{01} a_0 a_1^* - \mathrm{i} a_2 \sum_{k=0}^{2} \alpha_{2k} |a_k|^2.
\end{aligned} \tag{1}
$$

Following our previous work (Khan *et al* 1980), taking the second-order coupling factors to be equal to one, we obtain from (1) the corresponding real and imaginary parts as

$$
\begin{aligned}
\partial u_0/\partial t + \nu_0 u_0 &= \varepsilon u_1 u_2 \cos(\phi + \theta_{12}) - \varepsilon^2 u_0 \delta\nu_0', \\
\partial u_1/\partial t + \nu_1 u_1 &= \varepsilon u_0 u_2 \cos(\phi + \theta_{02}) - \varepsilon^2 u_1 \delta\nu_1', \\
\partial u_2/\partial t + \nu_2 u_2 &= \varepsilon u_0 u_1 \cos(\phi + \theta_{01}) - \varepsilon^2 u_2 \delta\nu_2',
\end{aligned} \tag{2}
$$

$$
\dot{\phi} = \Delta\omega - \varepsilon \frac{u_1 u_2}{u_0} \sin(\phi + \theta_{12}) - \varepsilon \frac{u_0 u_2}{u_1} \sin(\phi + \theta_{02}) - \varepsilon \frac{u_0 u_1}{u_2} \sin(\phi + \theta_{01}) - \varepsilon^2 \delta\omega'
$$

where

$$
\delta\omega' = \sum_{j=0}^{2} \beta_j u_j^2 \qquad \text{(the nonlinear frequency shift)},
$$

$$
\delta\nu_k' = -\sum_{k=0}^{2} \mathrm{Im}(\alpha_{jk}) u_k^2 \qquad \text{(the effective nonlinear dissipation)},
$$

$$
\beta_j = \mathrm{Re}\alpha_{j0} - \mathrm{Re}\,\alpha_{j1} - \mathrm{Re}\,\alpha_{j2},
$$

are the matrix elements of the coupling factors for the third-order terms taking normalisation of the amplitudes.

In order to solve the set of equations (2), we use the method of perturbation due to Coffey and Ford (1969), which is suitable for $\Delta\omega \neq 0$ and has limitations for $\Delta\omega = 0$. To separate the secular motion from the rapidly fluctuating motion, a solution is introduced of the form

$$
u_i = y_i + \sum_{n=1}^{\infty} \varepsilon^n F_i^{(n)}(\psi), \qquad i = 1, 2, \ldots, r, \tag{3}
$$

$$
\phi = \psi + \sum_{n=1}^{\infty} \varepsilon^n G_j^{(n)}(\psi), \qquad j = 1, 2, \ldots, s, \tag{4}
$$

$$\dot{y}_i = \sum_{n=0}^{\infty} \varepsilon^n a_i^{(n)}(y), \qquad i = 1, 2, \ldots, r, \tag{5}$$

$$\dot{\psi} = \Delta\omega + \sum_{n=1}^{\infty} \varepsilon^n b^{(n)}(y), \qquad j = 1, 2, \ldots, s. \tag{6}$$

Inserting (3), (4) in (2) and using (5) and (6), we finally obtain, after equating the powers of ε, the following set of equations (for details see Khan *et al* (1980)):

$$a_j^{(0)} + \nu_j y_j = 0, \qquad a_j^{(1)} = 0, \qquad b^{(1)} = 0, \tag{7a}$$

$$F_0^{(1)} = \frac{y_2 y_1}{\Delta\omega_0} \sin(\psi + \theta_{12} + \eta_0), \qquad F_1^{(1)} = \frac{y_2 y_0}{\Delta\omega_1} \sin(\psi + \theta_{02} + \eta_1),$$

$$F_2^{(1)} = \frac{y_0 y_1}{\Delta\omega_2} \sin(\psi + \theta_{01} + \eta_2), \tag{7b}$$

$$G^{(1)} = \frac{1}{\Delta\omega}\left(\frac{y_1 y_2}{y_0}\cos(\psi + \theta_{12}) + \frac{y_0 y_2}{y_1}\cos(\psi + \theta_{02}) + \frac{y_0 y_1}{y_2}\cos(\psi + \theta_{01})\right),$$

$$\tan\eta_j = \frac{\nu_j}{\Delta\omega_j}, \qquad \frac{1}{\Delta\omega_j} = \frac{1}{(\nu_j^2 + \Delta\omega^2)^{1/2}}. \tag{7c}$$

Also,

$$a_0^{(2)} = y_0 y_1^2 \left(\frac{1}{2\Delta\omega_2}\sin(\theta_{01} - \theta_{12} + \eta_2) + \frac{1}{2\Delta\omega}\sin(\theta_{01} - \theta_{12})\right)$$
$$+ y_0 y_2^2 \left(\frac{1}{2\Delta\omega_1}\sin(\theta_{02} - \theta_{12} + \eta_1) + \frac{1}{2\Delta\omega}\sin(\theta_{02} - \theta_{12})\right) - y_0 \delta\nu_0,$$

$$a_1^{(2)} = y_1 y_0^2 \left(\frac{1}{2\Delta\omega_2}\sin(\theta_{01} - \theta_{02} + \eta_2) + \frac{1}{2\Delta\omega}\sin(\theta_{01} - \theta_{02})\right)$$
$$+ y_1 y_2^2 \left(\frac{1}{2\Delta\omega_0}\sin(\theta_{12} - \theta_{02} + \eta_0) + \frac{1}{2\Delta\omega}\sin(\theta_{12} - \theta_{02})\right) - y_1 \delta\nu_1,$$

$$a_2^{(2)} = y_2 y_0^2 \left(\frac{1}{2\Delta\omega_1}\sin(\theta_{02} - \theta_{01} + \eta_1) + \frac{1}{2\Delta\omega}\sin(\theta_{02} - \theta_{01})\right)$$
$$+ y_2 y_1^2 \left(\frac{1}{2\Delta\omega_0}\sin(\theta_{12} - \theta_{01} + \eta_0) + \frac{1}{2\Delta\omega}\sin(\theta_{12} - \theta_{01})\right) - y_2 \delta\nu_2,$$

where

$$\delta\nu_j = -\sum_{k=0}^{2} \mathrm{Im}(\alpha_{jk}) y_k^2 \qquad (j = 0, 1, 2), \tag{8}$$

$$b^{(2)} = \frac{y_2^2 y_1^2}{2y_0^2}\left(\frac{\cos\eta_0}{\Delta\omega_0} - \frac{1}{\Delta\omega}\right) + \frac{y_0^2 y_2^2}{2y_1^2}\left(\frac{\cos\eta_1}{\Delta\omega_1} - \frac{1}{\Delta\omega}\right) + \frac{y_0^2 y_1^2}{2y_2^2}\left(\frac{\cos\eta_2}{\Delta\omega_2} - \frac{1}{\Delta\omega}\right)$$
$$- y_0^2 \left(\frac{\cos(\theta_{02} - \theta_{01} + \eta_1)}{2\Delta\omega_1} + \frac{\cos(\theta_{01} - \theta_{02} + \eta_2)}{2\Delta\omega_2} + \frac{\cos(\theta_{02} - \theta_{01})}{\Delta\omega}\right)$$
$$- y_1^2 \left(\frac{\cos(\theta_{01} - \theta_{12} + \eta_2)}{2\Delta\omega_2} + \frac{\cos(\theta_{12} - \theta_{01} + \eta_0)}{2\Delta\omega_0} + \frac{\cos(\theta_{12} - \theta_{01})}{\Delta\omega}\right)$$

$$-y_2^2\left(\frac{\cos(\theta_{02}-\theta_{12}+\eta_1)}{2\Delta\omega_1}+\frac{\cos(\theta_{12}-\theta_{02}+\eta_0)}{2\Delta\omega_0}+\frac{\cos(\theta_{12}-\theta_{02})}{\Delta\omega}\right)-\delta\omega$$

where $\delta\omega=\sum_{i=0}^{2}\beta_i y_i^2$.

$$G^{(2)}=\frac{1}{4\Delta\omega}\left(\frac{y_1^2}{\Delta\omega_2}\sin(2\phi+\theta_{01}+\theta_{12}+\eta_2)+\frac{y_2^2}{\Delta\omega_1}\sin(2\phi+\theta_{12}+\theta_{02}+\eta_1)\right.$$
$$+\frac{y_0^2}{\Delta\omega_2}\sin(2\phi+\theta_{01}+\theta_{12}+\eta_2)+\frac{y_2^2}{\Delta\omega_0}\sin(2\phi+\theta_{12}+\theta_{02}+\eta_0)$$
$$\left.+\frac{y_0^2}{\Delta\omega_1}\sin(2\phi+\theta_{01}+\theta_{02}+\eta_1)+\frac{y_1^2}{\Delta\omega_0}\sin(2\phi+\theta_{01}+\theta_{12}+\eta_0)\right)$$
$$-\left(\frac{y_1^2y_2^2}{y_0^2\Delta\omega_0}\sin(2\phi+2\theta_{12}+\eta_0)+\frac{y_0^2y_2^2}{y_1^2\Delta\omega_1}\sin(2\phi+2\theta_{02}+\eta_1)\right.$$
$$\left.+\frac{y_0^2y_1^2}{y_2^2\Delta\omega_2}\sin(2\phi+2\theta_{01}+\eta_2)\right)$$
$$-\frac{1}{\Delta\omega}\left(\frac{y_1^2y_2^2}{y_0^2}\sin 2(\phi+\theta_{12})+\frac{y_0^2y_2^2}{y_1^2}\sin 2(\phi+\theta_{02})+\frac{y_0^2y_1^2}{y_2^2}\sin 2(\phi+\theta_{01})\right)$$
$$-\frac{2}{\Delta\omega}\left(y_0^2\sin(2\phi+\theta_{02}+\theta_{01})+y_1^2\sin(2\phi+\theta_{12}+\theta_{01})\right.$$
$$\left.+y_2^2\sin(2\phi+\theta_{12}+\theta_{02})\right).$$

3. Solution of the secular motion in second order

From equations (5) and (8) we can obtain the second-order equations as

$$\dot{y}_0+\nu_0y_0=\varepsilon^2\left[y_0y_1^2\left(\frac{\sin(\theta_{01}-\theta_{12}+\eta_2)}{2\Delta\omega_2}+\frac{\sin(\theta_{01}-\theta_{12})}{2\Delta\omega}\right)\right.$$
$$\left.+y_0y_2^2\left(\sin\frac{(\theta_{02}-\theta_{12}+\eta_1)}{2\Delta\omega_1}+\frac{\sin(\theta_{02}-\theta_{12})}{2\Delta\omega}\right)-y_0\delta\nu_0\right],$$
$$\dot{y}_1+\nu_1y_1=\varepsilon^2\left[y_1y_0^2\left(\frac{\sin(\theta_{01}-\theta_{02}+\eta_2)}{2\Delta\omega_2}+\frac{\sin(\theta_{01}-\theta_{02})}{2\Delta\omega}\right)\right. \tag{9}$$
$$\left.+y_1y_2^2\left(\frac{\sin(\theta_{12}-\theta_{02}+\eta_0)}{2\Delta\omega_0}+\frac{\sin(\theta_{12}-\theta_{02})}{2\Delta\omega}\right)-y_1\delta\nu_1\right],$$
$$\dot{y}_2+\nu_2y_2=\varepsilon^2\left[y_2y_0^2\left(\frac{\sin(\theta_{02}-\theta_{01}+\eta_1)}{2\Delta\omega_1}+\frac{\sin(\theta_{02}-\theta_{01})}{2\Delta\omega}\right)\right.$$
$$\left.+y_2y_1^2\left(\frac{\sin(\theta_{12}-\theta_{01}+\eta_0)}{2\Delta\omega_0}+\frac{\sin(\theta_{12}-\theta_{01})}{2\Delta\omega}\right)-y_2\delta\nu_2\right].$$

3.1. Case of equal amplitudes

We consider the dissipative case when all $\nu_i=\nu$ and $y_i=y$ to obtain from equation (9)

$$\dot{y}+\nu y=\varepsilon^2y^3k \tag{10}$$

where

$$k = \frac{\sin(\theta_{01} - \theta_{12} + \eta)}{2\Delta\omega_k} + \frac{\sin(\theta_{02} - \theta_{12} + \eta)}{2\Delta\omega_k} + \frac{\sin(\theta_{01} - \theta_{12})}{2\Delta\omega} + \frac{\sin(\theta_{02} - \theta_{12})}{2\Delta\omega}$$

$$+ (\mathrm{Im}\ \alpha_{00} + \mathrm{Im}\ \alpha_{01} + \mathrm{Im}\ \alpha_{02})$$

$$= \frac{[\sin(\theta_{01} - \theta_{02} + \eta) + \sin(\theta_{12} - \theta_{02} + \eta)]}{2\Delta\omega_k} + \frac{[\sin(\theta_{01} - \theta_{02}) + \sin(\theta_{12} - \theta_{02})]}{2\Delta\omega}$$

$$+ \mathrm{Im}\ (\alpha_{10}) + \mathrm{Im}\ (\alpha_{11}) + \mathrm{Im}\ (\alpha_{12})$$

$$= \frac{[\sin(\theta_{02} - \theta_{01} + \eta) + \sin(\theta_{12} - \theta_{01} + \eta)]}{2\Delta\omega_k} + \frac{[\sin(\theta_{02} - \theta_{01}) + \sin(\theta_{12} - \theta_{01})]}{2\Delta\omega}$$

$$+ \mathrm{Im}\ (\alpha_{20}) + \mathrm{Im}\ (\alpha_{21}) + \mathrm{Im}\ (\alpha_{22}). \qquad (11)$$

The solution of the equation (10) is of the form

$$y(t) = [\gamma^2 + (t_1 - Bt)^2]^{-1/2} \qquad (12)$$

where

$$\gamma^2 = \varepsilon^2 k/\nu,$$

$$t_1 = (1/y(0))(1 - \gamma^2 y^2(0))^{1/2}, \qquad B = (\nu/y(0))(1 - \gamma^2 y^2(0))^{1/2}.$$

The solution (12) is a soliton and the amplitudes tend to zero for large times. The amplitude is limited to a maximum value

$$y_{\max} = 1/\gamma = \nu^{1/2}/\varepsilon k^{1/2}. \qquad (13)$$

For $\gamma = 0$ one has the expression for the amplitude

$$y(t) = y(0)/(1 - \nu t) \qquad (14)$$

and the time of explosion

$$t_\infty = 1/\nu. \qquad (15)$$

In the limit $\nu \simeq y(0)$ this is exactly the same as that obtained by Weiland and Wilhelmsson (1977). It is interesting to note that if γ is not equal to zero, $y(t)$ stays finite. The amplitudes are generally limited for physically realistic situations and the singular solution corresponds to the limiting case $\gamma = 0$.

3.2. Case of unequal amplitudes

We assume the effective nonlinear dissipations (Davydova *et al* 1975) are such that

$$\begin{aligned} &\mathrm{Im}\ (\alpha_{10}) = -\mathrm{Im}\ (\alpha_{01}) = \mu, \qquad \mathrm{Im}\ (\alpha_{21}) = -\mathrm{Im}(\alpha_{12}) = \mu, \\ &\mathrm{Im}\ (\alpha_{20}) = -\mathrm{Im}\ (\alpha_{02}) = \mu, \qquad \mathrm{Im}\ (\alpha_{ii}) = 0, \qquad i = 0, 1, 2. \end{aligned} \qquad (16)$$

With these assumptions, and further with $x_j = y_j^2$, $\nu \ll \Delta\omega$ (Fukai *et al* 1969), the constants of motion are obtained as

$$X_0 + X_1 + X_2 = P, \qquad X_0 X_1 X_2 = Q, \qquad (17)$$

where

$$x_j = X_j\,\mathrm{e}^{-2\nu t}, \qquad \tau = (1/2\nu)(1 - \mathrm{e}^{-2\nu t}).$$

Following our previous work (Khan *et al* 1980), we obtain

$$u_j(\tau) = \sin\theta \operatorname{sd}\left\{\frac{\sqrt{A}}{g-f} 2\varepsilon^2\left(\frac{\delta}{\Delta\omega}+\mu\right) \operatorname{cosec}\theta(\tau+\tau_j) \mid \sin^2\theta\right\}$$

and τ_j is given by

$$\tau_j = \frac{(g-f)}{\sqrt{A}} \frac{\sin\theta}{2\varepsilon^2(\delta/\Delta\omega+\mu)} \operatorname{sd}^{-1}\left(\frac{u_j(0)}{\sin\theta} \,\middle|\, \sin^2\theta\right) \tag{18}$$

where sd^{-1} is the Jacobian elliptic function.

$$X_j = \frac{f+gu_j}{1+u_j}, \qquad f = -\frac{\sqrt{3}+1}{2}\alpha, \qquad g = \frac{\sqrt{3}-1}{2}\alpha,$$

$$\alpha = (4Q)^{1/3}, \qquad \theta = 75°, \qquad A = \tfrac{3}{4}\sqrt{3}\,\alpha^4(\sqrt{3}+2),$$

$$\sin(\theta_{01}-\theta_{02}) = \sin(\theta_{12}-\theta_{01}) = \sin(\theta_{02}-\theta_{12}) = \delta.$$

When $u_j(\tau) = -1$, X_j tends to infinity and one can obtain the time of explosion given by

$$\sin\theta \operatorname{sd}\left[\frac{\sqrt{A}}{g-f} 2\varepsilon^2\left(\frac{\delta}{\Delta\omega}+\mu\right) \operatorname{cosec}\theta(\tau+\tau_j) \left|\sin^2\theta\right.\right] + 1 = 0. \tag{19}$$

When the initial conditions are such that $u_j(0) = \sin(\pi+\theta)\operatorname{sd}(l|\sin^2\theta)$ with l a constant, all the amplitudes will go to infinity at the same time and the time of explosion is given by

$$t_\infty = 3^{-1/4} \frac{[l - \operatorname{sd}^{-1}(\operatorname{cosec}\theta | \sin^2\theta)]}{2\varepsilon^2(\delta/\Delta\omega+\mu)}.$$

4. Discussion

In this paper an attempt has been made to study analytically the saturation of explosive instabilities by means of third-order nonlinear effects. The nonlinear effects considerably change the phase dynamics when the amplitudes are large and the saturation occurs as found both numerically and analytically. Numerical solutions of these problems have already been given by Weiland and Wilhelmsson. Hence our work complements the earlier work in the sense that the analytical results obtained are in agreement with the numerical solutions. After the saturation point the amplitudes quickly decrease. This effect is shown in figure 1 when all amplitudes are assumed to be equal in the presence of linear and nonlinear dissipative terms. If, however, the amplitudes are different the saturation peaks will occur repeatedly as functions of time (see figure 2). Oraevskii *et al* (1973a, b) obtained a complete analytical solution to the nonlinear coupled mode equations including a third-order frequency shift. They found a soliton solution, and with more general initial conditions repeated explosions were obtained which could be expressed in terms of elliptic functions. Weiland and Wilhelmsson (1973) and Weiland (1974) extended this investigation to include a linear dissipation and also an imaginary part of third-order frequency shift. They gave the numerical solutions for the general nonlinear system of equations containing third-order nonlinear terms, and also some analytical results concerning linear dissipation and asymptotic behaviour. These numerical results have been compared with those

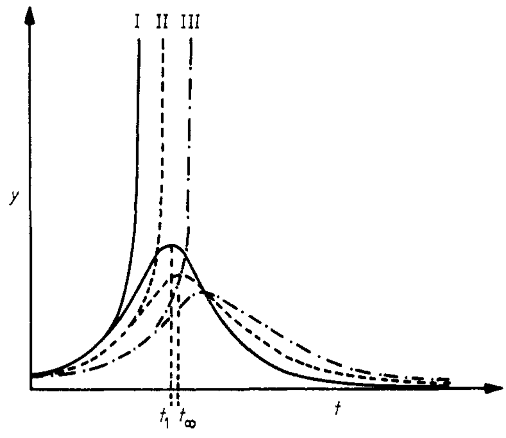

Figure 1. Curve I corresponds to the numerical solutions of Wilhelmsson, showing an unbounded solution ($\gamma = 0$) and the stabilised solution ($\gamma \neq 0$). Curve II corresponds to our analytical solution when $u(0) = 0.6$, $\nu = 0.6$, $\theta_{12} = 0$, $\theta_{02} = \pi/4$, $\theta_{01} = -\pi/4$, Im $\alpha_{ij} = 0.05$, showing an unbounded solution for $\gamma = \varepsilon k^{1/2}/\nu^{1/2} = 0$ and a stabilised solution for $\gamma \neq 0$. Curve III corresponds to our analytical solutions with only ν changed to 0.5.

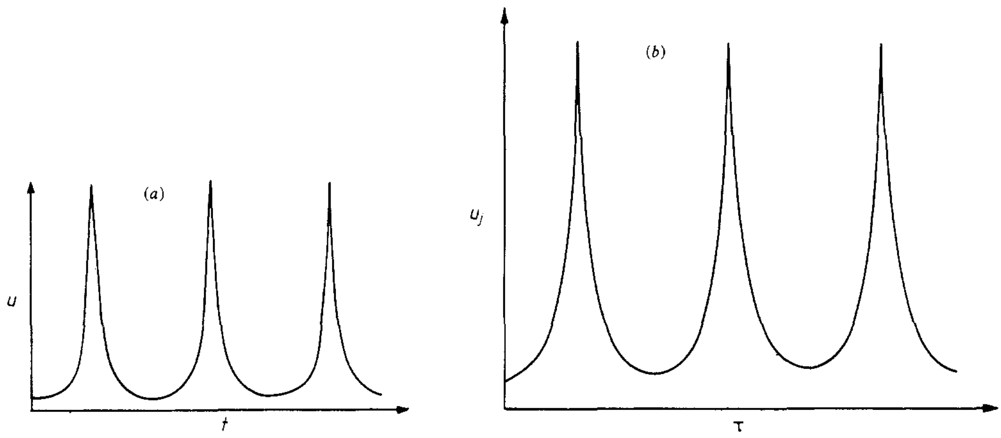

Figure 2. (a) corresponds to the numerical solution of Wilhelmsson, showing repeated stabilised explosions. (b) corresponds to our analytical solution, showing repeated stabilised explosions with $\theta_{01} = \pi/3$, $\theta_{12} = \pi$, $\theta_{02} = -\pi/3$.

obtained by us analytically, and found to be in agreement at least qualitatively, as can be seen in figures 1 and 2.

As shown above, the effect of complex coupling coefficients of second-order nonlinear terms, the influence of linear dissipation as well as third-order effects were taken into account to obtain analytically a soliton solution. In the presence of dissipation, second- and third-order conductivities have a real part, which means that coupling factors are complex when one must include a linear dissipation in the system. It is well known (see for example Fukai *et al* (1969)) that the effect of cubic-order terms causes an amplitude-dependent frequency shift. As the waves grow they shift out of resonance and $\Delta\omega$ ultimately becomes large. In the asymptotic limit of saturation, the values of the amplitudes are determined by introducing an effective coupling coefficient of the third-order coupling term, and the values are found to be independent of the initial conditions. It is shown that the nonlinear dissipation stabilises all amplitudes. Also the periodic solutions have been found in terms of the Jacobian elliptic function.

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